Complex numbers

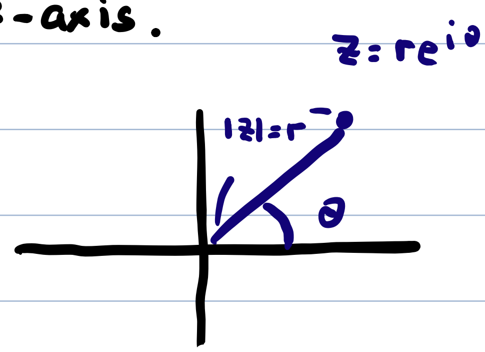
$z = a + bi \in \mathbb{C}$, $\quad \mathrm{Re}(z) = a \quad \mathrm{Im}(z) = b$.

Def **Magnitude**: $|z| = \sqrt{z \cdot \bar{z}} = \sqrt{a^2 + b^2}$

Argument: $\mathrm{Arg}(z) = \arctan(\mathrm{Re}\, z / \mathrm{Im}\, z)$

in $(-\pi, \pi]$ $\quad$ = angle between z and the positive z-axis.

- $e^{i\theta} = \cos\theta + i \sin\theta$



Def **Polar form**: $z = r \cdot e^{i\theta}$

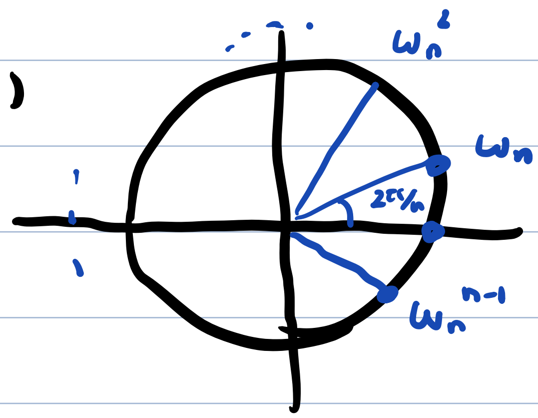
$|z| = r \qquad \mathrm{Arg}(z) = \theta$

(The n-th) **Roots of unity**: solutions of $z^n = 1$ in $\mathbb{C}$.

$\omega_n := e^{\frac{2\pi}{n} i}$, $\quad$ roots $= \{1, \omega_n, \omega_n^2, \cdots, \omega_n^{n-1}\}$ $\quad (n \geq 2)$

Prop $z^n = (z-1)(z-\omega_n) \cdots (z - \omega_n^{n-1})$

- $\omega_n \cdot \omega_n^2 \cdots \omega_n^{n-1} = (-1)^{n+1}$
- $1 + \omega_n + \omega_n^2 + \cdots + \omega_n^{n-1} = 0 \quad (n \geq 2)$



Complex functions

$f : \mathbb{C} \to \mathbb{C}$, $z = x + yi$

f also have a real and imaginary part.

$$f(z) = \underbrace{\mathrm{Re}(f(z))}_{u(x,y)} + \underbrace{\mathrm{Im}(f(z))}_{v(x,y)} \cdot i$$

$u, v : \mathbb{R}^2 \to \mathbb{R}$

$$f(z) = u(x,y) + i\, v(x,y)$$

e.g $f(z) = z^2 = (x+iy)^2 = (x^2 - y^2) + (2xy)i$

$u(x,y) = x^2 - y^2$

$v(x,y) = 2xy$

Def (Important functions) $z = re^{i\theta}$, $\theta \in (-\pi, \pi]$

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2}$$

$$\mathrm{Log}(z) = \ln r + i\theta$$
$$= \ln|z| + i\,\mathrm{Arg}(z)$$

Derivatives

Def $f : \mathbb{C} \to \mathbb{C}$ is differentiable at $z \in \mathbb{C}$ if the limit

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists.

$\left((z^n)' = nz^{n-1}, \quad (\sin z)' = \cos z, \quad \cdots \right)$

e.g $f(z) = \bar{z}$ is not differentiable at $z = 0$.

$h = a + bi \quad \bar{h} = a - bi$

$$\frac{f(h) - f(0)}{h} = \frac{\bar{h}}{h} = \begin{cases} 1 & b = 0 \\ -1 & a = 0 \end{cases}$$

Def $f : \mathbb{C} \to \mathbb{C}$ is a holomorphic function at $z \in \mathbb{C}$ if there is a nbhd of z s.t f is differentiable on V.

$z = x + iy$

Thm $f : \mathbb{C} \to \mathbb{C}$, $\quad f(z) = u(x,y) + i\,v(x,y)$ is holomorphic at z if u and v satisfy the following Cauchy-Riemann equation :

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{at } (x,y).$$

In this case, v is called a harmonic conjugate of u.

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}$$

Thm f is holomorphic on U $\Rightarrow$ f is analytic on U

↳ diffb ∞-ly many times & can be represented by a power series

Integrations

Def C : a curve in $\mathbb{C}$ parametrized by

$$z(t) = x(t) + i y(t) \quad a \leq t \leq b$$

$f : \mathbb{C} \to \mathbb{C}$ continuous

The path integral of f along C is

$$\int_C f(z)\, dz = \int_a^b f(z(t))\, z'(t)\, dt$$

Thm $F, f : \mathbb{C} \to \mathbb{C}$ $F'(z) = f(z)$ on $U \subseteq \mathbb{C}$.

If C is a curve in U from z_1 to z_2, then

$$\int_C f(z)\, dz = f(z_2) - f(z_1)$$

Thm (Cauchy) If $f : \mathbb{C} \to \mathbb{C}$ is a holomorphic function on $\mathbb{C}$ and C is a closed curve. Then

$$\int_C f\, dz = 0$$

Def $f: \mathbb{C} \to \mathbb{C}$. f is holomorphic on $\overset{\text{open}}{U} \setminus \{p\}$. Then p is a

removable singularity: if there is $\tilde{f}: \mathbb{C} \to \mathbb{C}$ holomorphic on U

s.t. $\tilde{f}(z) = f(z)$ on $U \setminus \{p\}$

(an n-th order)

pole: if there is a holomorphic function g on U s.t

$g(z) = (z-p)^n f(z)$ for $z \in U \setminus \{p\}$

essential singularity if it is neither removable nor a pole.

e.g $f(z) = \begin{cases} z & z \neq 0 \\ 1 & z = 0 \end{cases}$ 0 is a removable singularity

$g(z) = \frac{1}{z}$ $z \neq 0$ 0 is a 1st order pole

$h(z) = \cos\left(\frac{1}{z}\right)$ 0 is an essential singularity

Def $f: U \setminus \{p\} \to \mathbb{C}$ is holomorphic

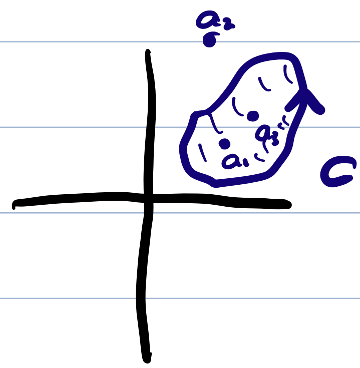
p is an n-th order pole. The **residue** of f at p is

$$\mathrm{Res}(f, p) = \frac{1}{(n-1)!} \lim_{z \to p} \frac{d^{n-1}}{dz^{n-1}} \left[(z-p)^n f(z) \right]$$

Thm $f: \mathbb{C} \to \mathbb{C}$, holomorphic on $U \setminus \{\overbrace{a_1, \cdots, a_k}^{\text{poles}}\}$. C is a closed curve in U w/ positive orientation (counter-clockwise)

$$\oint_C f(z)\,dz = 2\pi i \sum \mathrm{Res}(f, a_k)$$

a_k is inside of C



<u>Def</u> A <u>Laurent expansion</u> for $f: \mathbb{C} \to \mathbb{C}$ at p :

$$f(z) = \sum_{n=-\infty}^{\infty} C_n (z-p)^n$$

<u>e.g</u> $f(z) = \dfrac{5}{(1-z)(2i-z)}$ $\quad 1 < |z| < 2$

① partial fraction: $f(z) = \dfrac{1+2i}{2i-z} - \dfrac{1+2i}{1-z}$

② Use $\displaystyle\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ for $|z| < 1$:

$$\frac{1}{2i-z} = \frac{1}{2i} \cdot \frac{1}{1-(z/2i)} = \frac{1}{2i} \sum_{n=0}^{\infty} (z/2i)^n \quad \text{for } |z|<2$$

$$\frac{1}{1-z} = -\frac{1}{z} \cdot \frac{1}{1-1/z} = -\frac{1}{z} \sum_{n=6}^{\infty} \frac{1}{z^n} = -\sum_{n=1}^{\infty} \frac{1}{z^n} \quad |z|>1$$

$$\therefore f(z) = (1+2i) \cdot \left(\frac{1}{2i} \sum_{n=6}^{\infty} \left(\frac{z}{2i}\right)^n - \frac{1}{z} \sum_{n=1}^{\infty} \frac{1}{z^n} \right)$$